Arithmetic and Geometric Progressions (AP/GP) Summary

1. Arithmetic Progression (AP) :

- characterized by a common difference d

nth term $T_n = a + (n-1)d$, where a is the first term of the series

Sum to n terms $S_n = \frac{n}{2}(a + l)$, where l denotes the last term of the series of n terms

or $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ (to be used if value of l is not readily available)

2. Geometric Progression (GP) :

- characterized by a common ratio r

nth term $T_n = ar^{n-1}$, where a is the first term of the series

Sum to n terms $S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$

Note: If r is negative, then the sequence comprises terms alternating between positive and negative ones. Also, if $|r| < 1,$ then the series is a **convergent** one, ie $T_n \to k$ (k is a real value) as $n \to \infty$, and it possesses a **sum to infinity** given by $S_\infty = \frac{a}{1-r}$.

3. Techniques relevant to problem solving:

(i) <u>Forming an equation relating three terms of a sequence.</u>

If x, y and z are consecutive terms of an AP, then we have $y - x = z - y$

If x, y and z are consecutive terms of a GP, then we have $\frac{z}{y} = \frac{y}{x}$

(ii) <u>Recognising that any uninterrupted part of a sequence will retain the main characteristics of its parent sequence:</u>

Example: A descending geometric series has first term a and the common ratio r is positive. The sum of the first 5 terms is twice the sum of terms from the 6th to 15th inclusive.

While it is not erroneous to state that $S_5 = 2(S_{15} - S_5) \Rightarrow \frac{a(r^5-1)}{r-1} = 2\left[\frac{a(r^{15}-1)}{r-1} - \frac{a(r^5-1)}{r-1}\right]$,

it would be far more elegant to use the understanding that the 6th to 15th (10 terms altogether) terms of the GP can be construed as a separate GP series with first term $= ar^{6-1} = ar^5$ but with the same common ratio r as the parent series. Hence, we have

$$\frac{a(r^5-1)}{r-1} = 2\left[\frac{ar^5(r^{10}-1)}{r-1}\right],$$

which would render future developments of any solution a tad more efficient.

(iii) The bank interest problem (and its parallel storylines):

Example: At the end of a month, a customer owes a bank \$1500. In the middle of the month, the customer pays \$$x$ to the bank where x<1000, and at the end of the month the bank adds interest at a rate of 4% of the remaining amount still owed. This process continues every month until the money owed is repaid in full.

(i) Find the value of x for which the customer still owes \$1500 at the start of every month.

(ii) Find the value of x for which the whole amount owed is paid off exactly after the second payment.

(iii) Show that the value of x for which the whole amount owed is paid off exactly after the (n+1)th payment is given by

$$x = \frac{1500r^n(r-1)}{r^{n+1}-1}, \text{ where } r = 1.04$$

SOLUTIONS :

(i) 1500=(1500-x)(1.04) $\Rightarrow x = \$57.69$ (shown)

(ii) After 1st payment of \$$x$, the amount owed= (1500 - x)(1.04)

Therefore, (1500-x)(1.04)=x $\Rightarrow x = \$764.71$ (shown)

(iii) After the second payment of \$$x$, amount owed at beginning of 3rd month is

[(1500-x)1.04 $-x$] (1.04)=1500(1.04)^2 $-1.04^2x - 1.04x$

After the second payment of \$x, amount owed at beginning of 4th month is

$$[1500(1.04)^2 - 1.04^2 x - 1.04x - x](1.04)$$

$$=1500(1.04)^3 - 1.04^3 x - 1.04^2 x - 1.04x$$

After nth payment of \$ x, the amount still owed at the beginning of the ($n+1$)th month

$$=1500(1.04)^n - 1.04^n x - 1.04^{n-1} x - .... - 1.04^2 x - 1.04x$$

$$=1500r^n - x(r + r^2 + ... + r^n) \quad \text{where } r = 1.04$$

At the ($n+1$)th payment,

$$x = 1500r^n - x(r + r^2 + ... + r^n)$$

$$x(1 + r + r^2 + ... + r^n) = 1500r^n$$

$$\Rightarrow x(\frac{r^{n+1} - 1}{r - 1}) = 1500r^n \Rightarrow x = \frac{1500r^n (r-1)}{r^{n+1} - 1} \text{ (shown)}$$

Strategies to note in the above solution:

(a) Consolidation of values is highly discouraged when seeking out patterns in expressions , eg for (iii) of the immediate above problem, leave $1500(1.04)^2$ in its raw form, and do not simplify it as 1622.4.

(b) Exercise patience and write a few iterations applicable to various time frames, ie the outstanding amounts say after 1 month, 2 months, 3 months, etc or 1 year, 2 years, 3 years etc depending on the question context. From there, attempt to identify a trend/pattern within these formulations.

(c) Be acutely aware of the exact occurrences of events in reference to the timeline of the question, eg whether interest is appended at the beginning or end of a designated period, whether payment is made in the middle or the end of the period etc.

<u>(iv) Transforming a GP into an AP (or vice versa):</u>

This usually includes index operations or inclusion of logarithmic changes to every existing term of the original series.

Example: A geometric series $\{x_n\}$ has first term a and common ratio r. The sequence of numbers $\{y_n\}$ satisfies the relation $y_n = \log_3 x_n$ for $n \,\varepsilon\, Z,\ \ n \geq 1.$

(i) If the product of x_5 and x_{16} is 81, find the value of $\sum_{k=1}^{20} \log_3 x_k$.

(ii) Show that $\{y_n\}$ is an arithmetic sequence.

SOLUTIONS :

(i) $x_5 = ar^4,\ x_{16} = ar^{15}$

$x_5 x_{15} = 81 \Rightarrow a^2 r^{19} = 81$

$\sum_{k=1}^{20} \log_3 x_k = \log_3 x_1 + \log_3 x_2 + \log_3 x_3 + \ldots\ldots\ldots + \log_3 x_{20}$

$= \log_3 (x_1 x_2 x_3 \ldots\ldots\ldots x_{20})$

$= \log_3 [(x_1 x_{20})(x_2 x_{19}) \ldots\ldots\ldots (x_{10} x_{11})]$

$= \log_3 [(81)(81) \ldots\ldots\ldots (81)]$

$= \log_3 81^{10} = 10\log_3 81 = 10(4) = 40$(shown)

(**Note**: $x_1 x_{20} = x_2 x_{19} = \ldots\ldots\ldots = x_{10} x_{11} = a^2 r^{19} = 81$)

(ii) $y_n - y_{n-1} = \log_3 x_n - \log_3 x_{n-1} = \log_3 (\frac{x_n}{x_{n-1}}) = \log_3 r$, which is a constant.

Hence, $\{y_n\}$ is an arithmetic sequence.(shown)